Section 9.4

Fourier Series and LTI Systems

- Recall that a LTI system $\mathcal{H}$ with impulse response h is such that $\mathcal{H}\{z^n\} = H(z)z^n$, where $H(z) = \sum_{n=-\infty}^{\infty} h(n)z^{-n}$. (That is, complex exponentials are *eigensequences* of LTI systems.)
- Since a complex sinusoid is a *special case* of a complex exponential, we can reuse the above result for the special case of complex sinusoids.
- For a LTI system $\mathcal{H}$ with impulse response h and a complex sinusoid $e^{j\Omega n}$ (where Ω is real),

$$\mathcal{H}\{e^{j\Omega n}\} = H(e^{j\Omega})e^{j\Omega n},$$

  where

$$H(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} h(n)e^{-j\Omega n}.$$

- That is, $e^{j\Omega n}$ is an *eigensequence* of a LTI system and $H(e^{j\Omega})$ is the corresponding *eigenvalue*.
- The function $H(e^{j\Omega})$ is *2π-periodic*, since $e^{j\Omega}$ is 2π-periodic.
- We refer to $H(e^{j\Omega})$ as the frequency response of the system $\mathcal{H}$.

- Consider a LTI system with input x, output y, and frequency response $H(e^{j\Omega})$.
- Suppose that the N-periodic input x is expressed as the Fourier series

$$x(n) = \sum_{k=0}^{N-1} a_k e^{jk\Omega_0 n}, \quad \text{where } \Omega_0 = \frac{2\pi}{N}.$$

- Using our knowledge about the *eigensequences* of LTI systems, we can conclude

$$y(n) = \sum_{k=0}^{N-1} a_k H(e^{jk\Omega_0}) e^{jk\Omega_0 n}.$$

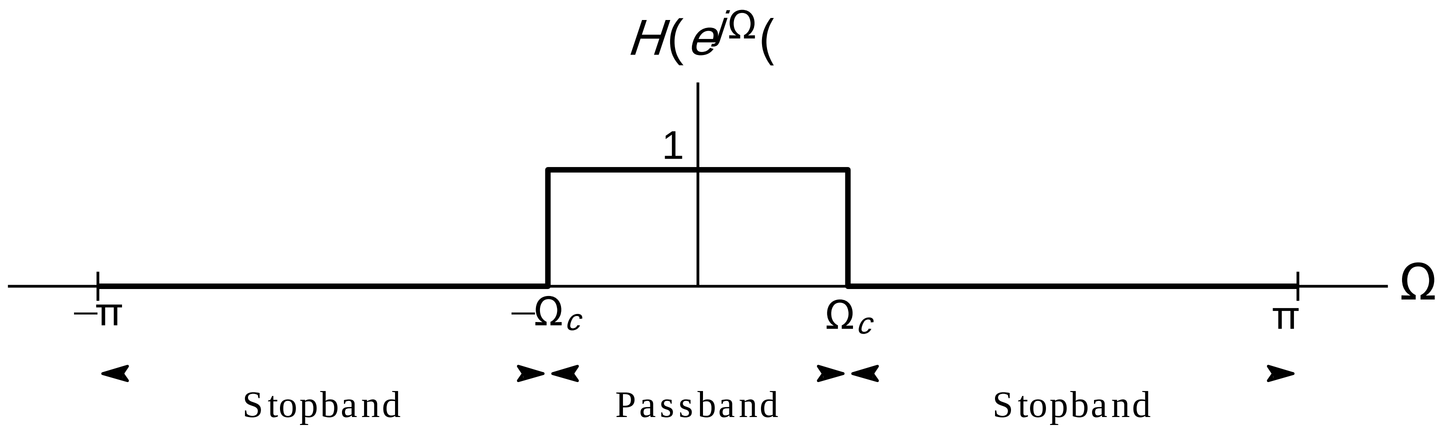
- Thus, if the input x to a LTI system is a Fourier series, the output y is also a Fourier series. More specifically, if $x(n) \overset{\text{DTFS}}{\longleftrightarrow} a_k$ then $y(n) \overset{\text{DTFS}}{\longleftrightarrow} H(e^{jk\Omega_0}) a_k$.
- The above formula can be used to determine the output of a LTI system from its input in a way that *does not require convolution*.

- In many applications, we want to *modify the spectrum* of a signal by either amplifying or attenuating certain frequency components.
- This process of modifying the frequency spectrum of a signal is called filtering.
- A system that performs a filtering operation is called a filter.
- Many types of filters exist.
- Frequency selective filters pass some frequencies with little or no distortion, while significantly attenuating other frequencies.
- Several basic types of frequency-selective filters include: lowpass, highpass, and bandpass.

- An ideal lowpass filter eliminates all baseband frequency components with a frequency whose magnitude is greater than some cutoff frequency, while leaving the remaining baseband frequency components unaffected.
- Such a filter has a *frequency response* of the form

$$H(e^{j\Omega}) = \begin{cases} 1 & \text{if } |\Omega| \leq \Omega_c \\ 0 & \text{if } \Omega_c < |\Omega| \leq \pi, \end{cases}$$

  where Ω_c is the cutoff frequency.
- A plot of this frequency response is given below.

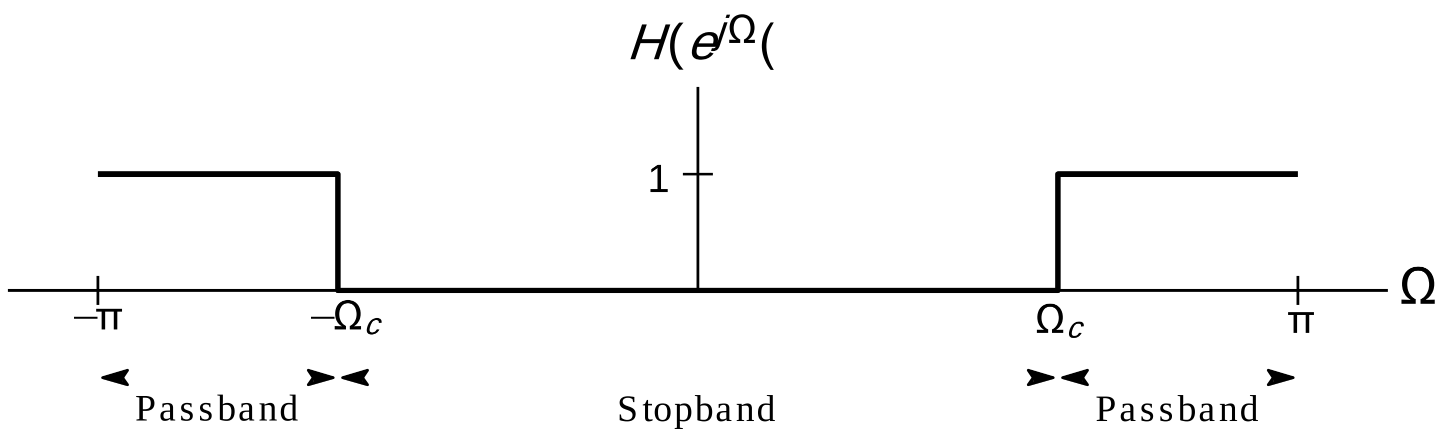


- An ideal highpass filter eliminates all baseband frequency components with a frequency whose magnitude is less than some cutoff frequency, while leaving the remaining baseband frequency components unaffected. Such a
- filter has a *frequency response* of the form

$$H(e^{j\Omega}) = \begin{cases} 1 & \text{if } \Omega_c < |\Omega| \le \pi \\ 0 & \text{if } |\Omega| \le \Omega_c, \end{cases}$$

where Ω_c is the cutoff frequency.

- A plot of this frequency response is given below.

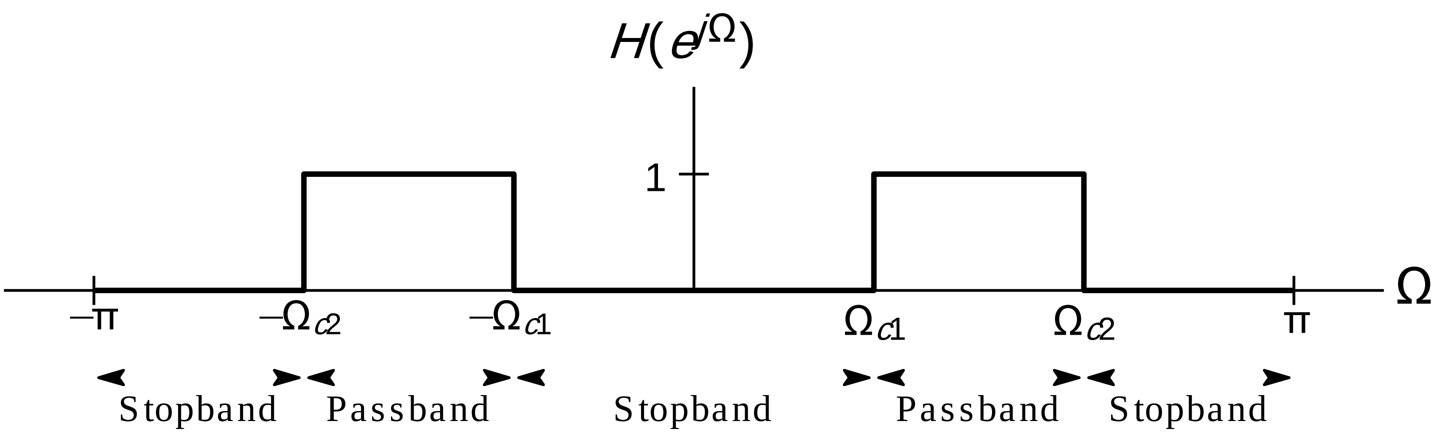


- An ideal bandpass filter eliminates all baseband frequency components with a frequency whose magnitude does not lie in a particular range, while leaving the remaining baseband frequency components unaffected.
- Such a filter has a *frequency response* of the form

$$H(e^{j\Omega}) = \begin{cases} 1 & \text{if } \Omega_{c1} \le |\Omega| \le \Omega_{c2} \\ 0 & \text{if } |\Omega| < \Omega_{c1} \text{ or } \Omega_{c2} < |\Omega| < \pi, \end{cases}$$

where the limits of the passband are Ω_{c1} and Ω_{c2}.

- A plot of this frequency response is given below.



Part 10

Discrete-Time Fourier Transform (DTFT(

- Fourier series provide an extremely useful representation for periodic signals.
- Often, however, we need to deal with signals that are not periodic. A
- more general tool than the Fourier series is needed in this case. The
- Fourier transform can be used to represent both periodic and aperiodic signals.
- Since the Fourier transform is essentially derived from Fourier series through a limiting process, the Fourier transform has many similarities with Fourier series.

Section 10.1

Fourier Transform

- The (DT) Fourier series is an extremely useful signal representation.
- Unfortunately, this signal representation can only be used for periodic sequences, since a Fourier series is inherently periodic.
- Many signals are not periodic, however.
- Rather than abandoning Fourier series, one might wonder if we can somehow use Fourier series to develop a representation that can also be applied to aperiodic sequences.
- By viewing an aperiodic sequence as the limiting case of an N-periodic sequence where $N \to \infty$, we can use the Fourier series to develop a more general signal representation that can be used for both aperiodic and periodic sequences.
- This more general signal representation is called the (DT) Fourier transform.

- The Fourier transform of the sequence x, denoted $\mathcal{F}\{x\}$ or X, is given by

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n}.$$

- The preceding equation is sometimes referred to as Fourier transform analysis equation (or forward Fourier transform equation.(
- The inverse Fourier transform of X, denoted $\mathcal{F}^{-1}\{X\}$ or x, is given by

$$x(n) = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j\Omega n} d\Omega.$$

- The preceding equation is sometimes referred to as the Fourier transform synthesis equation (or inverse Fourier transform equation). As a matter
- of notation, to denote that a sequence x has the Fourier transform X, we write $x(n) \overset{\text{DTFT}}{\longleftrightarrow} X(\Omega).($
- A sequence x and its Fourier transform X constitute what is called a Fourier transform pair.

Section 10.2

Convergence Properties of the Fourier Transform

- For a sequence x, the Fourier transform analysis equation (i.e., $X(\Omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n}$) converges *uniformly* if

$$\sum_{k=-\infty}^{\infty} |x(k)| < \infty$$

(i.e., x is *absolutely summable*).

- For a sequence x, the Fourier transform analysis equation (i.e., $X(\Omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n}$) converges in the *MSE sense* if

$$\sum_{k=-\infty}^{\infty} |x(k)|^2 < \infty$$

(i.e., x is *square summable*).

- For a bounded Fourier transform X, the Fourier transform synthesis equation (i.e., $x(n) = \frac{1}{2\pi}\int_{2\pi} X(\Omega) e^{j\Omega n} d\Omega$) will always converge, since the integration interval is finite.

Section 10.3

Properties of the Fourier Transform